References


A Matrix Key Distribution Scheme

can include up to \( b(\sqrt{N} - 2) \) extra bits in their session key. These extra bits are already available in their common key set but not used in our previous versions. In this case, according to the security proof, an extra number of \( \sqrt{N} - 2 \) colluding nodes not on the common key line might be needed, and at least \( \sqrt{N} - t \) extra such nodes are needed to attack their mutual communications. This is a big gain with little extra effort.

Note if a one-way function is used to compose a session key, it can ensure that attacking the session key itself does not help in attacking the common keys. When the one-way function is symmetric with respect to the dictionary order, it is not necessary to run the common keys through it in a fixed order, which speeds up the algorithm.

The security results are derived on the assumption that there is one node at each point on the communication map. They may become stronger if empty points are allowed. In fact, empty points could be specially allocated to enhance security, for example, to protect some vital nodes, or to segregate notes which have unclean records.

Or alternatively, it can be arranged that a node only receives the common keys it would use to talk to those nodes to which it is allowed to communicate. The little extra work by the key server, transparent to the concerned nodes, is to find out the allowed keys for a node and replace the other keys by random bits, before sending them off. This little effort gives an extra security feature.

We can easily generalize to multi-map multi-line versions, which allow different compromises between security and node key set length. There is little loss in choosing \( N = m_1 m_2 \), a product of two different prime numbers, but the extra complications seem not worthwhile.

A concept of logical positions of nodes could be introduced. Now the position of a node is not simply its physical address, but is assigned and can be changed by the key server. This has the advantage that to compromise a session key between a pair of nodes, a different group of colluding nodes is needed when logical relations change. This makes insider attacks more difficult.

The frequency of key change is based on the security level wanted. Normally, the whole or part of the key map is changed less frequently; the values of \( a \)'s, \( b \)'s, and the communication map are changed the least frequently. As fewer bits are to be generated and changed than in conventional schemes they can be changed more frequently.

Finally, in the multi-map version it can be arranged that some key maps are issued by other servers so that no single server can eavesdrop. The total amount of key traffic remains unchanged.
tables and no more than \( t - 1 \) reciprocals are needed. These reciprocals can be held in a table to speed up the calculation. Much of the above still holds when \( m = 2^n \).

Since communicating nodes construct a session key from the common keys, it is sufficient if every key on the key map is very short, because there are enough bits in common from which to construct the session key. Thus the number of bits that the key server has to generate and distribute could be very small. Suppose \( b \) is the common key length and the session key is required to be 64 bits long, then in the multi-line version, let \( t = 9 \) and \( b = 1 \), every pair of nodes has at least 72 key bits in common while in the multi-map version, let \( t = 16 \) and \( b = 2 \), the number of common key bits is at least 64. Of course making the total number of key bits larger by increasing \( t \) or \( b \) results in a higher level of security.

We summarize a comparison of the multi-line and the multi-map version. \( N \), \( b \), and \( t \) have the same meaning as before.

<table>
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<th>Multi-line version</th>
<th>Multi-map version</th>
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<td>Server generated key bits</td>
<td>( bN )</td>
<td>( btN )</td>
</tr>
<tr>
<td>Node stored key bits</td>
<td>( bt\sqrt{N} )</td>
<td>( 2bt\sqrt{N} )</td>
</tr>
<tr>
<td>Common key bits</td>
<td>( bt(t - 1) )</td>
<td>( 2bt )</td>
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<tr>
<td>Time to find common keys</td>
<td>( O(t^2) )</td>
<td>( O(t) )</td>
</tr>
<tr>
<td>Minimum colluding nodes</td>
<td>( \lceil t/3 \rceil^* )</td>
<td>( t )</td>
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* if \( \{c_i\} \) forms a super-increasing sequence

## 7 Security Considerations

When \( b \) is very small, we have to take into account the colluders' ability to perform exhaustive search. For example, in the multi-map version, suppose \( m \) colluders can search up to \( s \) bits. To compromise a session key it is required that

\[
2bm + s \geq 2bt, \quad \text{i.e.} \quad m \geq t - s/2b
\]

Thus the threshold is lowered from \( t \) to \( t - s/2b \). When the amount of key storage or transmission is fixed, i.e. \( b \times t \) is fixed, a smaller \( b \) increases the threshold. This also increases \( t \) and thus the time complexity to compute session keys. Another way to look at this, is, that since security requires \( 2bt > 2bm + s \) and the amount of key storage or transmission is proportional to \( bt \), a smaller \( b \) reduces the storage and transmission.

Keys generated and used are not necessarily equal in length. In fact, longer keys could be allocated to certain points to enhance the security of certain important nodes. As a more specific example, in the multi-map version, if all nodes who have a common key line on the communication map require higher mutual security, they
There are four cases to consider, i.e., whether the common key lines are rows or columns on the $p$th and the $q$th. The first case is that they are columns on the $p$th and the $q$th. Thus $x_A = x_B$ and $x'_A = x'_B$. Solving these equations we get $(a_p - a_q)(j - v) = 0$. Because $a$'s are distinct, thus $j = v$ which further results in $i = u$. This says that $A$ and $B$ are the same node, a contradiction. The other three cases are similar to this one.

Property 6 Two nodes have in common either $2t$ or $2t + \sqrt{N} - 2$ distinct keys.

Proof. If $A$ and $B$ have a common key line on a key map, they have $\sqrt{N}$ common keys on this map. According to the previous property, they do not have common key lines on any other key maps, so they have 2 common keys on each other map. Thus the total is $\sqrt{N} + 2(t - 1) = 2t + \sqrt{N} - 2$. If they never have a common key line, they have in total $2t$ common keys.

To simplify the following results and the proofs, we assume that when $A$ and $B$ have a common key line, they only use two of the $\sqrt{N}$ common keys selected by the solutions given above.

Property 7 When a pair of nodes communicate, any other node can have at most two of the common keys shared between the pair.

Proof. Let $C$ be a different node. From Property 5, $C$ can have at most one common line with $A$ and one with $B$ on all maps. So $C$ can have at most have two keys which are used between $A$ and $B$. 

Property 8 To compromise the key between two nodes, at least $t$ other colluding nodes are needed.

Proof. Considering that two nodes have at least $2t$ common keys, this property is a straightforward corollary of the previous one.

In general this lower bound is also an upper bound because there is always a group of $t$ nodes who when colluding together can compromise the session key. However, as stated before, more colluding nodes may be required if empty points could be specially allocated on the communication map to enhance security.

6 Enhancing Performance

We can simplify the computation in the multi-line version by choosing $c_i$'s as consecutive numbers so that the computation cycle can simply step through the
matrix in which each point \((i, j)^t\) corresponds to a key \(k_{ij}^l\). The key set given to \(n_{ij}\) is

\[
K_{ij} = \{k_{xy}^l | x = i + a_i j, \ or, \ y = i + b_i j, \ mod(m)\} \\
l = 1, 2, \ldots, t
\]

where a’s and b’s are all distinct. \(K_{ij}\) consists of key rows or columns on the key maps, called key lines.

Algorithm \textit{generate} generates \(tm^2\) random keys and puts one on each point on every key map. Then it selects \(K_{ij}\) by the definition and sends it to the node at the address \(P_{ij}\).

Algorithm \textit{compute} takes a pair of addresses on the communication map as input, say \(P_{ij}\) and \(P_{us}\), and finds out the common keys \(k_{pq}^l\) where

\[
p = i + a_i j \text{ and } q = u + b_i v, \ or, \ p = u + a_i v \text{ and } q = i + b_i j, \ l = 1, 2, \ldots, t
\]

It then uses the common keys to compose a session key, possibly using a one-way function.

Note that if \(a_i = 0, 2, \ldots, 2t - 2\) \text{ and } \(b_i = a_i + 1\) then the above computation reduces to selecting keys from consecutive key lines by an index starting with value \(u\) and being stepped by \(v\) modulo \(m\).

The above is easier to understand if \(m\) is a prime; but in fact other values of \(m\) work. For example, to make implementation easier and faster, it may be desirable to make \(m\) a power of 2; however, the following analysis may not hold in this case.

\textbf{Property 4} The key server has to generate \(tN\) keys and send to each node \(t\sqrt{N}\) keys. The time needed to set up a session key is the time complexity of algorithm \textit{compute} and is \(O(t)\).

\textbf{Property 5} Assume \(m\) is prime. If two nodes have a common key line on a key map, they do not have any common key lines on any other key maps.

\textbf{Proof.} All the calculations are done \(mod(m)\). Assume A and B have two common key lines, one on the \(p\)th key map and one on the \(q\)th. A’s and B’s key lines on the \(p\)th key map are indicated respectively by the row and column indices

\[
x_A = i + a_p j, \ y_A = i + b_p j \text{ and } x_B = u + a_p v, \ y_B = u + b_p v
\]

Indices of key lines on the \(q\)th key map are

\[
x'_A = i + a_q j, \ y'_A = i + b_q j \text{ and } x'_B = u + a_q v, \ y'_B = u + b_q v
\]
Property 2 Assuming \( m \) is prime and \( c_p \neq c_q \) when \( p \neq q \), two different nodes have in common either \( t(t - 1) \) or \( \sqrt{N} + (t - 1)(t - 2) \) distinct keys.

Proof. Every two non-parallel lines always meet at exactly one point because \( m \) is a prime. Observe that two nodes cannot have more than one common line, and it is only at the two points where the nodes sit that more than two key lines can intersect, thus if they do not have a common line they have exactly \( t(t - 1) \) intersecting points; otherwise, they have

\[
m + (t - 1)(t - 2) = \sqrt{N} + (t - 1)(t - 2)
\]
such points. \( \square \)

Property 3 Assuming \( m \) is prime and \( c_i \)'s are distinct, then for a particular pair of nodes, there exist a number of groups of \( t - 1 \) other nodes who, when colluding together, will be able to compromise the session key between the pair of nodes. There are a number of groups of \( t \) colluding nodes that can compromise A's session keys with any node.

Proof. There exists a group of \( t - 1 \) nodes each of which has a distinct key line in common with node A and covers another distinct key on the \( t \)th line used in the session. This group is able to compromise A's particular key for that session. A group of colluding nodes, one on each of the \( t \) lines through A, can compromise all A's communications. \( \square \)

We have not derived a satisfactory lower bound of the minimum number of colluding nodes needed to compromise a session key. The only result is that if \( \{c_i\} \) is a super-increasing sequence and \( m \) is sufficiently large, then to compromise the key between two nodes, at least \( \lceil t/3 \rceil \) colluding nodes are needed. However, we strongly doubt that this lower bound could be reached in most cases or the constraints on \( c_i \)'s are necessary. Therefore we skip the tedious proof here. Note more colluding nodes may be needed when empty points are allowed on the communication map.

5 Multi-Map Version

This version is achieved by generating more key maps but still allocating two key lines on each map to every node.

Assume there are \( N \) nodes where \( N = m^2 \). A communication map is defined as an \( m \times m \) matrix in which each point \( i, j \) has an address \( P_{ij} \) and corresponds to a node \( n_{ij} \) where \( i, j = 1, 2, \ldots, m \). The \( l \)th key map is also defined as an \( m \times m \)
A Matrix Key Distribution Scheme

Assume there are $N$ nodes where $N = m^2$. A communication map is defined as an $m \times m$ matrix on which each point has an address $P_{ij}$ and corresponds to a node $n_{ij}$, where $i, j = 1, 2, \ldots, m$. A key map is also defined as an $m \times m$ matrix where point $i, j$ corresponds to a key $k_{ij}$. The key set sent to node $n_{ij}$ is

$$K_{ij} = \{k_{xy} | y - j + c_i(x - i) = 0 \mod (m)\}$$

$l = 1, 2, \ldots, t$ and $c_p \neq c_q$ when $p \neq q$

The key set is a set of $t$ lines on the key map all passing through point $i, j$. These keys can be stored in $t$ tables where each entry is indexed by the value $x - i$. This makes the calculation of common keys easier as we will see later.

Algorithm generate generates $m^2$ random keys and puts one on each point on the key map. It calculates $K_{ij}$ from the simple definition, sends it to the node at the address $P_{ij}$.

Algorithm compute takes a pair of addresses on the communication map as input, say $P_{ij}$ and $P_{uv}$, solves $t(t - 1)$ linear equation groups each of which has the form

$$y - j + c_p(x - i) = 0 \mod (m)$$

$$y - v + c_q(x - u) = 0 \mod (m)$$

$p, q = 1, 2, \ldots, t$ and $p \neq q$

The solution is in fact very simple:

$$x - i = (c_q(i - u) + v - j)/(c_p - c_q) \mod (m)$$

The solutions $(x, y)$ are positions on the communication map of keys that nodes $n_{ij}$ and $n_{uv}$ have in common. There is no need to calculate $y$ because $p, q$ determines the key line table and the $x - i$ is the index of the key. Compute then composes a session key from these keys, possibly using a one-way function. Using a table look up for the needed reciprocals will speed up the calculation.

When $m$ is chosen to be a prime, each equation group has exactly one solution. Other choices of $m$ also work. For example, to let implementation be easier and faster, it may be desirable to make $m$ a power of 2. In this case, when $(c_p - c_q)$ is odd, there is exactly one solution; so if we choose half $c_i$'s to be even and half odd, there are at least $t^2/2$ common keys between each pair of nodes. When it is even and the numerator in the solution equation is odd, there is no solution and when both are even there might be none, one, or more than one solution. These extra keys can be used to enhance security, although it is unlikely to be worth the complication.

Property 1 The key server has to generate $N$ keys and send to each node $t\sqrt{N}$ keys. The time needed to set up a session key is the time complexity of algorithm compute which is $O(t^2)$. 
Two properties are interesting. First, two nodes can start a session virtually without delay. Second, the storage requirement is reduced by a square root factor. The key server will generate \( N \) keys in total instead of \( N(N - 1)/2 \) and each node receives and stores \( 2\sqrt{N} \) keys instead of \( N \). However, this basic scheme is weak in that if \( A \) and \( B \) are on the same line or column, any node on the same line or column could compromise the session because it shares the same common keys used between \( A \) and \( B \). When \( A \) and \( B \) are not on the same line or column, the situation is better as two correctly positioned colluding nodes are needed to compromise the session key.

3 Principle

Assume there are \( N \) nodes, \( n_i, i = 1, 2, \ldots, N \). Associated with each node \( n_i \) is a set of keys \( K_i \) and a published address \( P_i \).

Given the number of nodes \( N \), a random number \( R \) as a seed, a set of constraints \( C \), an algorithm \( \text{generate}(N, R, C) \) generates the sets of random keys \( K_i \) such that
\[
\forall i, j, k, i \neq j \neq k : K_i \cap K_j \neq \emptyset \quad \text{and} \quad K_i \cap K_j \neq K_i \cap K_k
\]

Given two addresses and the key set of node \( n_i \), an algorithm \( \text{compute}(P_i, P_j, K_i) \) derives a session key \( k_{ij} \). Using a fixed order of the common keys independent of \( i \) or \( j \), or a symmetric one-way function, we can arrange that
\[
\text{compute}(P_i, P_j, K_i) = \text{compute}(P_j, P_i, K_j) \quad \text{i.e.} \quad k_{ij} = k_{ji}
\]

A key server generates and distributes all the keys. When \( A \) wants to communicate to \( B \), the protocol is

1. \( A \) finds out \( B \)'s address and \( k_{AB} = \text{compute}(P_A, P_B, K_A) \).
2. \( A \) encrypts with \( k_{AB} \) and sends to \( B \) specifying source \( A \).
3. \( B \) calculates \( k_{AB} = k_{BA} = \text{compute}(P_B, P_A, K_B) \).
4. \( B \) decrypts using \( k_{AB} \) and continues if the decryption is successful.

Based on this principle, we extend the basic scheme in two directions. This results in the multi-line version and the multi-map version.

4 Multi-Line Version

This version is achieved by allocating more key lines to each node instead of only two as in the basic scheme.
the symmetric key generation scheme [1]. They both use expensive algebraic codes. They are also common in that the threshold of the number of colluding nodes required to compromise a single key equals the threshold to compromise all the keys. In our scheme, the thresholds are different. And it is possible to give higher security to particular groups of nodes. The matrix scheme is also more cost-effective. For example, analysis of [3] shows that the number of possible nodes cannot be greater than the number of key bits sent to each node. In our scheme, the former can be significantly greater than the latter. This means that to maintain a network of the same size and of at least the same security level, our scheme needs much less transmission and storage.

We first illustrate the new idea with a simple example. Then we describe the general principle and two versions of the matrix scheme. We analyze the security and performance and discuss some possible extensions to enhance them.

2 A Basic Scheme

Assume there are $N$ nodes, where $N = m^2$. Each node is assigned a position $i, j$, and is denoted as $n_{ij}$. Similarly, there are $N$ keys denoted as $k_{ij}$.

A key server generates the keys at random and gives node $n_{ij}$ a set of keys which consists of all the keys that are either on the same row or column as the node, $K_{ij} = \{k_{xy} \mid x = i, \text{ or }, y = j\}$. When node A ($n_{ij}$) wants to communicate to B ($n_{uv}$), it simply finds out B's position $u, v$ and uses the keys $k_{iv}$ and $k_{uj}$ which are common between A and B to compose a session key, e.g., just concatenates the two keys. See the diagram below for an illustration.

![Fig. 1. The Key Map.](image-url)


1 Introduction

The effectiveness of any cryptographic system is highly dependent on the techniques used for selecting, handling, and protecting the keys. Key distribution is a major problem in an environment where a large number of nodes communicate with each other. In this paper we assume node-to-node encryption rather than link-to-link encryption which is considered unsuitable in an open system environment [6]. We do not address the issue of enhancing security by using host master keys, secondary keys, or key-encryption keys [4].

Proposed so far are three major key distribution schemes. The first is to use a private-key cipher system and arrange that each pair of nodes share a different secret key. Thus \( N \) nodes require \( N(N - 1)/2 \) keys be generated and distributed by a secure key manager. Each node has to maintain \( N \) keys for all possible communications. This is known as the \( N^2 \) problem.

The second is to use a public-key system [2]. Each node selects its own key pair \((E, D)\) and publishes \( E \). When node A wants to communicate with node B, A encrypts the message using key \( E_B \) and sends the ciphertext to B. The ciphertext can only be decrypted using the secret key \( D_B \). This avoids the \( N^2 \) problem but public-key encryptions and decryptions are expensive and slow. A usual variation is to establish a private session key between each pair of nodes using a public key when they start communication. A drawback of such schemes is that there is a considerable delay before nodes can start the session. Each node has to cache one session key for each other node it wants to talk to.

The third scheme is to use an authentication server to set up session keys [5]. When a pair of nodes want to communicate, they first authenticate themselves to the server. The server then generates and distributes a session key. Authentication protocols can be based either on a private-key cipher or on a public-key cipher. The delay here is even greater than in the previous scheme.

All these come from the concept of a secret key which people assume is unique and kept completely secret. The fundamental idea of the scheme we propose here, in contrast to the concept above, is to let each node have a set of keys of which it shares a distinct subset with every other node. A key server generates the keys and distributes them as often as required. Upon receiving the keys, nodes can immediately start communicating to any other node. Our scheme requires a total of \( O(N) \) instead of \( N(N - 1)/2 \) keys. Each node needs to hold only \( O(\sqrt{N}) \) keys. Moreover, as we shall see later, the keys can be as short as 8 bits or even less. The scheme introduces an extra risk that it is possible for nodes to collude and compromise the session keys of other nodes. However, it can be arranged that the minimum number of colluding nodes required is acceptable.

There are two other key distribution schemes, the predistribution scheme [3] and
A Matrix Key Distribution Scheme

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Abstract. A new key distribution scheme is presented. It is based on the distinctive idea that lets each node have a set of keys of which it shares a distinct subset with every other node. This has the advantage that the numbers of keys that must be generated is proportional to the number of nodes. Moreover, two nodes can start a session with virtually no delay. The scheme suits an environment where there is a certain level of trust among the insiders. The security property to an outsider remains identical to that of other existing schemes. Two versions of the scheme are given. Analysis of security and performance shows it is a practical solution to some key distribution problems.

Key words. Communication Security, Private-key Cipher, Session Key, Key Distribution.

1To be published in the Journal of Cryptology, 1989.